ISRAEL JOURNAL OF MATHEMATICS, Vol. 59, No. 2, 1987

THE CONJUGACY CHARACTER OF S_n TENDS TO BE REGULAR

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ABSTRACT

Consider the regular and the conjugacy characters of S_n as vectors in Euclidean space, with the standard inner product. As n grows, the angle between them tends to zero and the ratio of their lengths tends to one. The two characters have therefore asymptotically similar decompositions into irreducible components.

1. Introduction

Any finite group G has the following two familiar representations on its group algebra

$$\mathbf{C}G = \left\{ \sum_{g\in G} \alpha_g g \;\middle|\; \alpha_g \in \mathbf{C} \right\}:$$

(i) The *(left) regular representation* R:

$$R_g(h) = gh \qquad (g, h \in G).$$

(ii) The *conjugacy representation* C:

$$C_g(h) = ghg^{-1} \qquad (g, h \in G).$$

Denote the corresponding characters by χ_R, χ_C.

The vector space of complex class functions on G has an inner product:

$$(1.1) \qquad \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Received March 23, 1987

THEOREM. *Let $\chi_R^{(n)}$, $\chi_C^{(n)}$ be the regular and the conjugacy characters of the symmetric group S_n. Then*

(i) $\lim_{n\to\infty} \| \chi_R^{(n)} \| / \| \chi_C^{(n)} \| = 1$,

(ii) $\lim_{n\to\infty} \langle \chi_R^{(n)}, \chi_C^{(n)} \rangle / \| \chi_R^{(n)} \| \cdot \| \chi_C^{(n)} \| = 1$.

Notice that the irreducible characters of a group form an orthonormal basis for the space of class functions with the inner product (1.1) [2, p. 223]. The above result may therefore be interpreted as follows:

"For large n, the conjugacy and regular representations of S_n have essentially the same multiplicities of irreducible components."

This statement is informative since we know relatively little about the above multiplicities for the conjugacy representation of S_n (unlike the case with the regular representation). A sample result is that those multiplicities are all positive, for $n \neq 2$ [3].

2. Proof of Theorem

Let G be a finite group. Denote (for $g \in G$):

$$C_G(g) = \{h \in G \mid gh = hg\} \quad (\textit{centralizer of } g),$$

$$\hat{g} = \{h^{-1}gh \mid h \in G\} \quad (\textit{conjugacy class of } g),$$

$$\hat{G} = \{\hat{g} \mid g \in G\}.$$

Now define

$$f(G) = \sum_{\hat{g} \in \hat{G}} \frac{1}{|\hat{g}|}.$$

The result stated above obviously follows from the following two propositions.

PROPOSITION 1. *For any finite group G,*

$$\frac{\| \chi_R \|}{\| \chi_C \|} = \frac{\langle \chi_R, \chi_C \rangle}{\| \chi_R \| \cdot \| \chi_C \|} = \frac{1}{\sqrt{f(G)}}.$$

PROPOSITION 2.

$$\lim_{n\to\infty} f(S_n) = 1.$$

(The proof of Proposition 2 is due to the first author.)

PROOF OF PROPOSITION 1. Obviously (denoting by e the unit element of G):

$$\chi_R(g)=\begin{cases}|G|, & g=e,\\ 0, & g\neq e,\end{cases}$$

$$\chi_C(g)=|C_G(g)|=|G|/|\hat{g}|.$$

Therefore, for any character χ of G:

$$\langle\chi_R,\chi\rangle=\frac{1}{|G|}\chi_R(e)\overline{\chi(e)}=\chi(e)$$

(which is the degree of χ). In particular

$$\langle\chi_R,\chi_R\rangle=\langle\chi_R,\chi_C\rangle=|G|.$$

On the other hand

$$\langle\chi_C,\chi_C\rangle=\frac{1}{|G|}\sum_{g\in G}\chi_C(g)^2=\frac{1}{|G|}\sum_{g\in G}\frac{|G|^2}{|\hat{g}|^2}$$

$$=|G|\sum_{g\in G}\frac{1}{|\hat{g}|^2}=|G|\sum_{\hat{g}\in\hat{G}}\frac{1}{|\hat{g}|}=|G|f(G).$$

Therefore

$$\frac{\|\chi_R\|}{\|\chi_C\|}=\frac{\langle\chi_R,\chi_C\rangle}{\|\chi_R\|\cdot\|\chi_C\|}=\frac{1}{\sqrt{f(G)}}$$

as claimed. ■

PROOF OF PROPOSITION 2. It is well-known that if a permutation $g\in S_n$ has r_i cycles of length i ($i=1,2,\ldots$) then its conjugacy class $\hat{g}$ consists of all permutations with the same sequence of cycle-lengths, and

$$|\hat{g}|=\frac{n!}{\prod_{i\geq 1}(r_i!\,i^{r_i})} \tag{2.1}$$

[e.g. 4, p. 67].

Inspection shows that

$$|\hat{g}| = \binom{n}{r_1} \cdot \frac{(n-r_1)!}{\prod_{i \geqq 2} (r_i!\, i^{r_i})} = \binom{n}{r_1} |\hat{g}'|$$

where $g' \in S_{n-r_1}$ has no 1-cycles (i.e. fixed points), but otherwise has the same cycle lengths as g.

Therefore, denoting

$$S'_n = \{g \in S_n \mid g \text{ has no fixed point}\},$$

$$\hat{S}'_n = \{\hat{g} \mid g \in S'_n\},$$

$$f'(S_n) = \sum_{\hat{g}' \in \hat{S}'_n} \frac{1}{|\hat{g}'|},$$

we obtain

$$f(S_n) = \sum_{r_1=0}^{n} \frac{f'(S_{n-r_1})}{\binom{n}{r_1}} = \sum_{r=0}^{n} \frac{f'(S_r)}{\binom{n}{r}}$$

(by convention, $f(S_0) = f'(S_0) = 1$).

We want to bound $f'(S_n)$, through finding $\min\{|\hat{g}'| : g' \in S'_n\}$. First assume that $g \in S'_n$ has maximal cycle-length $m > 4$. Obtain $g_1 \in S'_n$ by breaking each m-cycle of g into an $(m-2)$-cycle and a 2-cycle. Then, by (2.1)

$$\begin{aligned}\frac{|\hat{g}|}{|\hat{g}_1|} &= \frac{(r_2+r_m)!\, 2^{r_2+r_m}(r_{m-2}+r_m)!\,(m-2)^{r_{m-2}+r_m}}{r_2!\, 2^{r_2} r_{m-2}!\,(m-2)^{r_{m-2}} r_m!\, m^{r_m}} \\ &= \binom{r_2+r_m}{r_m}\binom{r_{m-2}+r_m}{r_m} r_m! \left(\frac{2m-4}{m}\right)^{r_m} \\ &> 1\end{aligned}$$

since $r_m \geqq 1$, $2m-4 > m$.

Similarly, if $g \in S'_n$ has maximal cycle-length $m = 4$, then by breaking each 4-cycle into two 2-cycles we get

$$\frac{|\hat{g}|}{|\hat{g}_1|} = \frac{(r_2+2r_4)!\, 2^{r_2+2r_4}}{r_2!\, 2^{r_2} r_4!\, 4^{r_4}} = \binom{r_2+2r_4}{r_2}\frac{(2r_4)!}{r_4!} > 1$$

since $r_4 \geqq 1$.

We conclude so far that $g \in S_n'$ with minimal $|\hat{g}|$ must have $r_1 = r_4 = r_5 = \cdots = 0$.

Now let $g \in S_n'$ have r_i cycles of length i ($i = 2, 3$). Then $2r_2 + 3r_3 = n$, and denoting

$$p = \frac{3r_3}{n} \qquad (0 \leqq p \leqq 1)$$

we get (defining, by continuity, $x \ln x \,|_{x=0} = 0$)

$$\begin{aligned} \ln|\hat{g}| &= \ln \frac{n!}{r_2!\, 2^{r_2} r_3!\, 3^{r_3}} \\ &= n \ln n + O(n) - [r_2 \ln r_2 + O(r_2) + r_3 \ln r_3 + O(r_3)] \\ &= \left(1 - \frac{1-p}{2} - \frac{p}{3}\right) n \ln n + O(n) \geqq \tfrac{1}{2} n \ln n + O(n). \end{aligned}$$

The number of conjugacy classes in S_n is $p(n)$, the number of partitions of n for which we know [1, p. 316]

$$\ln p(n) \leqq K\sqrt{n} \qquad (K = \pi\sqrt{\tfrac{2}{3}})$$

Therefore we obtain the bound

$$\text{(2.2)}\quad \ln f'(S_n) \leqq \ln|\hat{S}_n'| - \ln \min_{g \in \hat{S}_n'} |\hat{g}| \leqq O(\sqrt{n}) - \tfrac{1}{2} n \ln n - O(n) \leqq -n$$

for n large enough, e.g. $n \geqq N$.

Fix $k \geqq N$. Then, for $n \geqq 2k$

$$f(S_n) = \sum_{r=0}^{n} \frac{f'(S_r)}{\binom{n}{r}} \leqq \sum_{r=0}^{k-1} \frac{f'(S_r)}{\binom{n}{r}} + \sum_{r=k}^{n-k} \frac{e^{-r}}{\binom{n}{k}} + \sum_{r=n-k+1}^{n} e^{-r} \leqq \sum_{r=0}^{k-1} \frac{f'(S_r)}{\binom{n}{r}} + O(n^{-k}).$$

Since, by definition,

$$f(S_n) \geqq \sum_{r=0}^{k-1} \frac{f'(S_r)}{\binom{n}{r}} \qquad (n \geqq k-1)$$

we obtain (first for $k \geqq N$, and thus for any $k \geqq 1$)

(2.3)
$$f(S_n)=\sum_{r=0}^{k-1}\frac{f'(S_r)}{\binom{n}{r}}+O(n^{-k})\qquad (k\geqq 1).$$

In particular, for $k=1$, we get $f(S_n)=1+O(1/n)$. ∎

3. Remarks

For any $\varepsilon>0$, let us define

$$f_\varepsilon(S_n)=\sum_{\hat{g}\in\hat{S}_n}|\hat{g}|^{-\varepsilon},$$

$$f'_\varepsilon(S_n)=\sum_{\hat{g}\in\hat{S}'_n}|\hat{g}|^{-\varepsilon}.$$

From an ε-version of the bound (2.2) we may get the following generalization of the asymptotic expansion (2.3):

Corollary 1. *For any $\varepsilon>0$ and $k\geqq 1$,*

$$f_\varepsilon(S_n)=\sum_{r=0}^{k-1}f'_\varepsilon(S_r)\binom{n}{r}^{-\varepsilon}+O(n^{-\varepsilon k})$$

and in particular

$$\lim_{n\to\infty}f_\varepsilon(S_n)=1\qquad(\forall\,\varepsilon>0).$$

Note that for $\varepsilon=0$, $f_0(S_n)=|\hat{S}_n|=p(n)\to\infty$. For $\varepsilon=1$ we get Proposition 2.

The asymptotic expansion (2.3) also implies

Corollary 2. *For the above characters $\chi_R^{(n)}$, $\chi_C^{(n)}$ of S_n, define $\rho_n>1$ and $0<\phi_n<\pi/2$ by*

$$\|\chi_C^{(n)}\|=\rho_n\|\chi_R^{(n)}\|,$$

$$\langle\chi_R^{(n)},\chi_C^{(n)}\rangle=\|\chi_R^{(n)}\|\cdot\|\chi_C^{(n)}\|\cos\phi_n,$$

then

$$\rho_n=1+\frac{1}{n^2}+O(n^{-3}),$$

$$\phi_n = \frac{\sqrt{2}}{n} + O(n^{-2}).$$

PROOF. For $k = 3$, (2.3) gives

$$f(S_n) = 1 + \frac{2}{n^2} + O(n^{-3}).$$

Use $\rho_n^{-1} = \cos \phi_n = f(S_n)^{-1/2}$ (Proposition 1). ∎

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